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A minor-based characterization of matroid 3-connectivity

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For Geoff Whittle in recognition of his inspirational contributions to matroid theory

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ABSTRACT

It is well known that a matroid is 2-connected if and only if every 2-element set is contained in a circuit, or equivalently, a $U_{1,2}$-minor. This paper proves that a matroid is 3-connected if and only if every 4-element set is contained in a minor isomorphic to a wheel of rank 3 or 4; a whirl of rank 2, 3, or 4; or the relaxation of a rank-3 whirl. Some variants of this result are also discussed.

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1. Introduction

Matroid k-connectivity is typically defined in terms of a connectivity function and the absence of small separations. There is a familiar alternative characterization of 2-connectivity that can be stated quite plainly.

1.1. Proposition. *A matroid M is 2-connected if and only if every 2-element subset of $E(M)$ is contained in a circuit of M.*

This characterization can also be expressed in terms of minors.

1.2. Proposition. *A matroid M is 2-connected if and only if every 2-element subset of $E(M)$ is contained in a $U_{1,2}$-minor of M.*

These characterizations of 2-connectivity are succinctly written in terms of well-understood containment relations. However, no characterizations of this type for higher connectivity had been known.

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The following is the main result of this paper. Recall that $\mathcal{W}_r$ denotes the r-spoked wheel graph, $\mathcal{W}^r$ denotes the rank-r whirl, and Q_6 denotes the unique relaxation of a circuit-hyperplane in the rank-3 whirl.

1.3. Theorem. *A matroid M having at least four elements is 3-connected if and only if, for each 4-element subset X of $E(M)$, there is a minor N of M such that $X \subseteq E(N)$, and N is isomorphic to one of $\mathcal{W}^2$, $\mathcal{W}^3$, $\mathcal{W}^4$, $M(\mathcal{W}_3)$, $M(\mathcal{W}_4)$, or Q_6.*

There is a similar characterization of 3-connectivity in terms of 5-element sets. The next theorem specializes that result to binary matroids. Here $K_5 - e$ denotes the single-edge deletion of the graph K_5, and S_8 is the unique deletion of a non-tip element from the tipped rank-4 binary spike.

1.4. Theorem. *A binary matroid M having at least five elements is 3-connected if and only if, for each 5-element subset X of $E(M)$, there is a minor N of M such that $X \subseteq E(N)$, and N is isomorphic to one of $M(\mathcal{W}_3)$, $M(\mathcal{W}_4)$, $M(\mathcal{W}_5)$, $M(K_5 - e)$, $M^*(K_5 - e)$, $M(K_{1,2,3})$, $M^*(K_{1,2,3})$, or S_8.*

Section 2 contains some basic results that are needed in the proofs of Theorems 1.3 and 1.4. These proofs appear in Sections 3 and 4. In Section 5, there is a discussion of extending this type of characterization to a result in terms of k-element sets for any fixed $k \geqslant 4$. Explicit lists of matroids characterizing 3-connectivity in the k-subset case are not given, but a description of the largest matroids in these lists is provided.

The concluding remarks in Section 6 note a variation on the results proved in this paper that guarantees 3-connectivity in terms of a much weaker condition. Difficulties in obtaining characterizations of higher connectivity via minor containment are also discussed.

2. Preliminaries

The matroid notation and terminology used in this paper follow Oxley [3]. A matroid M *uses* a set X of elements if $X \subseteq E(M)$. For a positive integer n, the set $\{1, 2, \dots, n\}$ will be denoted by $[n]$.

The proofs of Theorems 1.3 and 1.4 rely on the following result, which is known as Bixby's Lemma [1] (see also [3, Lemma 8.7.3]). Recall that a k-separation (A, B) is *non-minimal* if $|A|, |B| > k$.

2.1. Lemma. *Let e be an element of a 3-connected matroid M. Then either $M\backslash e$ or M/e has no non-minimal 2-separations. Moreover, in the first case, $\operatorname{co}(M\backslash e)$ is 3-connected, while, in the second case, $\operatorname{si}(M/e)$ is 3-connected.*

The next lemma follows from Bixby's Lemma and is used in the proofs of Theorems 1.3 and 1.4.

2.2. Lemma. *Let M be a 3-connected matroid having more than k elements for some fixed $k \geqslant 2$. Let X be a k-element subset of $E(M)$. If no 3-connected proper minor of M uses X, then, for each $e \in E(M) - X$, there is a pair $\{x, y\} \subseteq X$ such that $\{e, x, y\}$ is a triangle or a triad of M.*

Proof. Suppose some element e of $E(M)$ is not in a triangle or a triad containing two members of X. By switching to the dual if necessary, we may assume, by Bixby's Lemma, that M/e has no non-minimal 2-separations. Each parallel class of M/e contains at most one member of X, so there is a proper minor of M isomorphic to $\operatorname{si}(M/e)$ that uses X. $\square$

Section 5 contains a proof that relies on the following result of Bixby and Coullard [2] (see also [3, Theorem 12.3.6]).

2.3. Theorem. *Let N be a 3-connected minor of a 3-connected matroid M with $|E(N)| \geqslant 4$. Suppose that $e \in E(M) - E(N)$ and M has no 3-connected proper minor that both uses e and has N as a minor. Then, for some (N_1, M_1) in $\{(N, M), (N^*, M^*)\}$, one of the following holds where $|E(M) - E(N)| = n$:*

(1) *$n = 1$ and $N_1 = M_1 \backslash e$;*
(2) *$n = 2$ and $N_1 = M_1 \backslash e/f$ for some element f; and N_1 has an element x such that $\{e, f, x\}$ is a triangle of M_1;*
(3) *$n = 3$ and $N_1 = M_1 \backslash e, g/f$ for some elements f and g; and N_1 has an element x such that M_1 has $\{e, f, x\}$ as a triangle and $\{f, g, x\}$ as a triad; moreover, $M_1 \backslash e$ is 3-connected;*
(4) *$n = 3$ and M_1 has a triad $\{e, f, g\}$ such that $N_1 = M_1 \backslash e, g/f = M_1 \backslash e, f, g$; moreover, N_1 has distinct elements x and y such that $\{e, g, x\}$ and $\{e, f, y\}$ are triangles of M_1; or*
(5) *$n = 4$ and $N_1 = M_1 \backslash e, g/f, h$ for some elements f, g, and h; and N_1 has an element x such that $\{e, f, x\}$ and $\{g, h, x\}$ are triangles of M_1, and $\{f, g, x\}$ is a triad of M_1; moreover, each of $M_1 \backslash e$, $M_1 \backslash e/f$, and $M_1 \backslash h/g$ is 3-connected.*

One direction of the equivalences in the main results of this paper is an easy consequence of the well-known persistence of separations through minors of a matroid.

2.4. Proposition. *Let $k \geqslant 4$ be a fixed integer. Suppose M is a matroid having at least k elements, and $\mathcal{N}$ is a nonempty set of 3-connected matroids, each having at least k elements. If, for each k-element subset X of $E(M)$, there is an $\mathcal{N}$-minor of M using X, then M is 3-connected.*

Proof. Suppose M is not 3-connected. Let (A, B) be a j-separation of M for some $j < 3$. Choose a k-subset X of $E(M)$ such that $|X \cap A|, |X \cap B| \geqslant j$. Then X is in no 3-connected minor of M. □

Observe that the converse of this proposition also holds if, for instance, the set $\mathcal{N}$ is taken to be all 3-connected matroids having at least k elements. Certainly the characterization obtained in this manner is of questionable value. The results in this paper concern the minimal lists needed to achieve these characterizations of 3-connectivity.

3. Matroid 3-connectivity in terms of 4-element sets

This section proves the main result of the paper.

Proof of Theorem 1.3. Note that the list of matroids given in the statement of the theorem is closed under duality since each of its members is self-dual. Other than $M(\mathcal{W}_3)$, $\mathcal{W}^3$, and Q_6, the 3-connected matroids with at least four but not more than six elements consist of uniform matroids of rank and corank at least two, and P_6, the unique relaxation of Q_6. Each 4-element subset of each of these matroids is contained in a $\mathcal{W}^2$-minor of the matroid. Thus the theorem holds when $|E(M)| \leqslant 6$.

Now assume that M has at least seven elements and that there is some four-element subset $X = \{a, b, c, d\}$ of $E(M)$ such that no 3-connected proper minor of M uses X. First observe the following.

3.1. Neither M nor M^* has a rank-2 flat containing more than three elements.

If M has such a flat Y containing X, then $M|X \cong \mathcal{W}^2$; a contradiction. If $X \not\subseteq Y$ then, for any $y \in Y - X$, the matroid $M \backslash y$ is 3-connected; another contradiction. By duality, 3.1 holds. Note that 3.1 and Lemma 2.2 together restrict the possible structure of M enough to reduce the proof to a finite case check.

The following is an immediate consequence of 3.1.

3.2. $3 \leqslant r(M) \leqslant |E(M)| - 3$.

3.3. If M has a triad, then $r(M) \geqslant 4$.

Let Y be a triad of M. As $|E(M) - Y| \geqslant 4$, it follows by 3.1 that $r(E(M) - Y) \geqslant 3$, so $r(M) \geqslant 4$.

By Lemma 2.2, duality, and relabeling, we may assume the following.

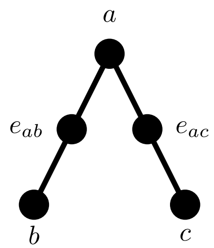


Fig. 1. A geometric representation of $M|(T_1 \cup T_2)$ when X is not a basis of M.

3.4. The set $\{e, a, b\}$ is a triangle of M for some $e \in E(M) - X$.

Next, we show that:

3.5. If, for some g in $E(M) - X - e$, there is a triad of M containing g and two elements of X, then M is isomorphic to $M(\mathcal{W}_4)$ or $\mathcal{W}^4$.

By 3.4 and orthogonality, the triad must be $\{g, a, b\}$ or $\{g, c, d\}$. In each case, it follows by Lemma 2.2, orthogonality, and 3.1 that every f in $E(M) - X - e - g$ is in a triangle with $\{c, d\}$ or is in a triad with $\{a, b\}$ or $\{c, d\}$. But by 3.1, $E(M) - X - e$ has at most three elements, namely at most one element in a triad with $\{a, b\}$, at most one element in a triad with $\{c, d\}$, and at most one element in a triangle with $\{c, d\}$. By 3.3 and duality, $|E(M)| \geqslant 8$. Hence $|E(M)| = 8$ and each element of X is in both a triangle and a triad. Thus if $x \in X$, then neither $M\backslash x$ nor M/x is 3-connected. By assumption, if $x \in E(M) - X$, then neither $M\backslash x$ nor M/x is 3-connected. Thus 3.5 holds by Tutte's Wheels-and-Whirls Theorem [4] (see also [3, Theorem 8.8.4]).

We may assume that each member of $E(M) - X$ is in a triangle with two elements of X. Let e_{xy} denote the unique element of $E(M) - X$ that is in a triangle with the pair $\{x, y\} \subseteq X$, if this element exists. Note that X spans M, and $M|X$ is a matroid on four elements not isomorphic to $U_{2,4}$ by 3.1, so $M|X$ is isomorphic to $U_{3,4}$, $U_{2,3} \oplus U_{1,1}$, or $U_{4,4}$.

As M has at least seven elements, we may assume by relabeling if necessary that M has triangles $T_1 = \{a, b, e_{ab}\}$ and $T_2 = \{a, c, e_{ac}\}$. A geometric representation for $M|(T_1 \cup T_2)$ is shown in Fig. 1. Suppose $M|X$ is isomorphic to $U_{3,4}$ or $U_{2,3} \oplus U_{1,1}$. Then d is in the plane of M spanned by $T_1 \cup T_2$. If d is on exactly one line spanned by two elements of $T_1 \cup T_2$, then M has a minor isomorphic to $\mathcal{W}^3$ using X. If d is on two such lines, then M has an $M(\mathcal{W}_3)$-minor using X. Otherwise, d lies on no such line, so M has a Q_6-minor that uses X.

Finally, suppose $M|X$ is isomorphic to $U_{4,4}$. Then d is not on the plane spanned by $T_1 \cup T_2$. As M has no 1-element or 2-element cocircuits, every element of X is in at least two triangles that each contain exactly one other element of X. Up to relabeling, assume that $\{e_{ab}, e_{ac}, e_{cd}, e_{bd}\} \subseteq E(M) - X$. Let $N = M|(X \cup \{e_{ab}, e_{ac}, e_{cd}, e_{bd}\})$. A geometric representation for N is shown in Fig. 2 where possibly $\{e_{ab}, e_{ac}, e_{cd}, e_{bd}\}$ is a circuit. Thus N is isomorphic to one of $\mathcal{W}^4$ or $M(\mathcal{W}_4)$. $\square$

Binary and graphic corollaries to Theorem 1.3 are immediate.

3.6. Corollary. *A binary matroid M having at least four elements is 3-connected if and only if, for each 4-element subset X of $E(M)$, there is a minor M that uses X and is isomorphic to $M(\mathcal{W}_3)$ or $M(\mathcal{W}_4)$.*

3.7. Corollary. *A graph G having no isolated vertices and at least four edges is simple and 3-connected if and only if, for each 4-element subset X of $E(G)$, there is a minor H of G such that $X \subseteq E(H)$ and H is isomorphic to a 3- or 4-spoked wheel graph.*

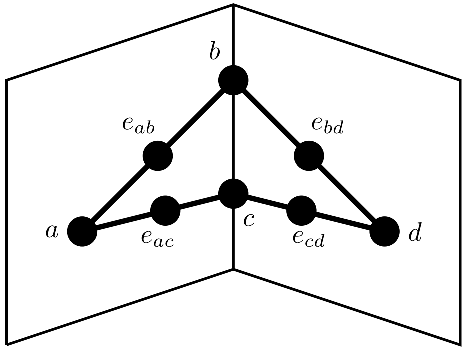


Fig. 2. A representation for the matroid N when X is a basis of M.

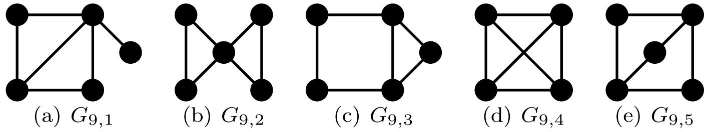


Fig. 3. Graphs whose cycle matroids are the $PG(3,2)$-complements of 9-element, simple, rank-4 binary matroids.

4. Binary matroid 3-connectivity in terms of 5-element sets

Observe that there is a 3-element version [3, Proposition 4.3.6] of Proposition 1.2.

4.1. Proposition. *A matroid M having at least three elements is 2-connected if and only if every 3-element subset of $E(M)$ is contained in a $U_{1,3}$- or $U_{2,3}$-minor of M.*

As noted in the Introduction, there is an analogous characterization of 3-connectivity in terms of five-element sets. In this section, Theorem 1.4, the binary restriction of this characterization, is proved.

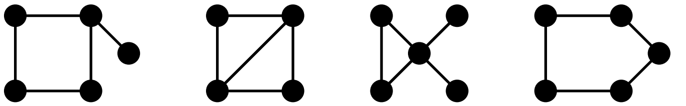
Proof of Theorem 1.4. Observe that the list of matroids in the statement of this theorem is closed under duality. Let M be a 3-connected binary matroid having at least five elements and at most eight elements. Suppose M is not isomorphic to a matroid in the list given in the theorem. Then M is isomorphic to F_7, F_7^*, or $AG(3,2)$. Every single-element deletion of F_7 is isomorphic to $M(\mathcal{W}_3)$, and every single-element deletion of $AG(3,2)$ is isomorphic to F_7^*. Thus, any set of five elements in M can be captured in an $M(\mathcal{W}_3)$-minor. Thus the theorem holds for matroids having at most eight elements.

Now suppose M is a simple binary matroid having exactly nine elements. Then M has rank 4 or 5. By duality, assume M is rank 4. View M as a restriction of $PG(3,2)$ and consider the complement of M in $PG(3,2)$; that is, consider $PG(3,2)\backslash E(M)$. This is a 6-element binary matroid and so is graphic. Thus the possibilities for M can be determined via consideration, up to 2-isomorphism, of all simple graphs on at most five vertices that have exactly six edges. These graphs are given in Fig. 3.

The following argument shows that the theorem holds for M as either M is isomorphic to $M(K_5 - e)$, or M has at least six distinct elements f such that $M\backslash f$ is isomorphic to $M(\mathcal{W}_4)$ or S_8.

The $PG(3,2)$-complement of $M(G_{9,5})$ is not 3-connected. The complement of $M(G_{9,1})$ has four single-element deletions isomorphic to $M(\mathcal{W}_4)$ and two single-element deletions isomorphic to S_8. The complement of $M(G_{9,2})$ is isomorphic to $M^*(K_{3,3})$, so each of its single-element deletions is isomorphic to $M(\mathcal{W}_4)$. The complement of $M(G_{9,3})$ is isomorphic to $M(K_5 - e)$. The complement of $M(G_{9,4})$ is the tipped rank-4 binary spike, so the deletion of any element other than the tip is S_8.

The 10-element, rank-4, simple binary matroids are $PG(3,2)$-complements of the cycle matroids of the graphs in Fig. 4. Moreover, each of the graphs in this figure is 2-isomorphic to a single-edge deletion of one of $G_{9,1}$, $G_{9,2}$, $G_{9,3}$, or $G_{9,4}$. Thus, each of the rank-4, simple, binary matroids having



(a) $G_{10,1}$ (b) $G_{10,2}$ (c) $G_{10,3}$ (d) $G_{10,4}$

Fig. 4. Graphs whose cycle matroids are the $PG(3,2)$-complements of 10-element, simple, rank-4 binary matroids.

ten elements is 3-connected. Of the graphs in Fig. 4, only $G_{10,1}$ has the property that an edge can be added to obtain a graph 2-isomorphic to $G_{9,5}$. It follows that the $PG(3,2)$-complement of $M(G_{10,1})$ is the only rank-4, simple, binary matroid with ten elements having a single-element deletion that is not 3-connected. Only three of its single-element deletions fail to be 3-connected, so the theorem holds for all rank-4 binary matroids having ten elements. Furthermore, the theorem holds for all rank-4 binary matroids having more than ten elements since it holds for every 10-element restriction of each of these matroids.

Thus M is a 3-connected binary matroid with $r(M), r^*(M) \geqslant 5$, and there is some subset $X = \{a,b,c,d,e\}$ of $E(M)$ such that no 3-connected proper minor of M uses X. Since M is binary, no rank-2 flat of M or M^* contains more than three elements. By Lemma 2.2, every element of $E(M) - X$ is in a triangle or a triad with two elements of X. By orthogonality, $E(M) - X$ cannot contain a four-element subset $\{x_1, x_2, y_1, y_2\}$ such that x_1 and x_2 are each in triangles containing two members of X, and y_1 and y_2 are each in triads containing two members of X; otherwise M has at most nine elements. We may assume, by duality, that there is at most one $y \in E(M) - X$ such that y is in a triad with exactly two elements of X.

First consider the case when there is an element y such that, without loss of generality, $\{y,a,b\}$ is a triad. Each element of $E(M) - X - y$ is in a triangle with a pair of elements in $X = \{a,b,c,d,e\}$ having an even intersection with $\{a,b\}$. There are at most four such pairs. Since M has at least ten elements, all these four possible triangles are present in M. Thus $|E(M)| = 10$, and $E(M)$ has a 6-element subset Z of rank 3. Evidently X spans $E(M) - y$, so X spans M. Hence $r(M) \leqslant 5$, so $r(M) = 5$. Then $r^*(E(M) - Z) = |E(M) - Z| + r(Z) - r(M) = 2$. This is a contradiction since $|E(M) - Z| = 4$, and no coline of the binary matroid M has more than three elements.

Now assume that each element of $E(M) - X$ is in a triangle with exactly two elements of X. Note that $r(X) = r(M) = 5$. Let $A = [I_5|D]$, where D is the X-fundamental circuit incidence matrix of M. The matrix obtained by appending the row $[1\,1\,1\,1\,1\,0\,0\,\ldots\,0]$ to A yields a $GF(2)$-representation of M. View this representation as the vertex–edge incidence matrix of a 3-connected simple graph G having six vertices and at least ten edges. The elements of X label the five edges incident with a vertex v in G. If the 5-vertex graph $G - v$ has a Hamilton cycle, then M has a restriction using X that is isomorphic to $M(\mathcal{W}_5)$. Since G is 3-connected, the graph $G - v$ is 2-connected. The unique subgraph-minimal graph on five vertices having at least five edges that is 2-connected but not Hamiltonian is $K_{2,3}$. If $G - v$ has a subgraph isomorphic to $K_{2,3}$, then M has a restriction containing X that is isomorphic to $M(K_{1,2,3})$. $\square$

The following result for graphs is an immediate consequence of Theorem 1.4.

4.2. Corollary. *A graph G having no isolated vertices and at least five edges is simple and 3-connected if and only if, for each 5-element subset X of $E(G)$, there is a minor H of G such that $X \subseteq E(H)$ and H is isomorphic to a 3-, 4-, or 5-spoked wheel graph; or $K_5 - e$, or its planar dual, the 3-prism; or $K_{1,2,3}$.*

5. Largest matroids characterizing 3-connectivity in terms of *k*-element sets

For each $k \geqslant 4$, let $\mathcal{N}_k$ be the set of 3-connected matroids M having a k-element subset X such that no 3-connected proper minor of M uses X. The following result is a straightforward extension of Proposition 2.4.

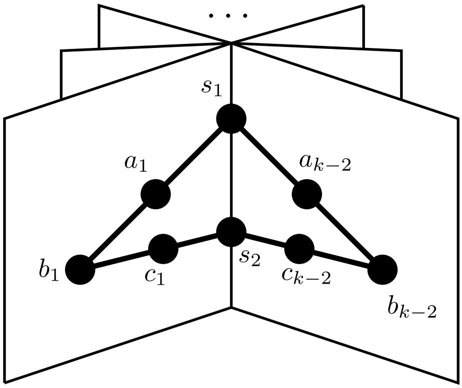


Fig. 5. A visualization of a rank-k codex with $k-2$ pages.

5.1. Proposition. *For each $k \geqslant 4$, a matroid M with at least k elements is 3-connected if and only if every k-element subset of $E(M)$ is contained in an $\mathcal{N}_k$-minor of M.*

The definition of $\mathcal{N}_k$ means that the last result fails if $\mathcal{N}_k$ is replaced by any proper subset. It is now not difficult to see that $\mathcal{N}_k$ is the unique minimal set of matroids characterizing 3-connectivity in terms of k-element sets. By the next result, members of $\mathcal{N}_k$ have at most $3k-4$ elements. Hence, $\mathcal{N}_k$ is certainly finite.

5.2. Proposition. *For each $k \geqslant 4$, each member of $\mathcal{N}_k$ has at most $3k-4$ elements.*

Proof. The proposition holds when $k=4$ by Theorem 1.3. Suppose that $k>4$ and that the proposition holds for $\mathcal{N}_{k-1}$. Let M be a member of $\mathcal{N}_k$. Let X be a k-element subset of $E(M)$ that is used by no 3-connected proper minor of M. Let $e \in X$. Then there is an $\mathcal{N}_{k-1}$-minor N of M using $X-e$. By Theorem 2.3, $|E(M)-E(N)| \leqslant 4$. However, since the matroid $M_1 \backslash h/g$ is 3-connected in case (5) of Theorem 2.3, we tighten that bound to $|E(M)-E(N)| \leqslant 3$. Since $|E(N)| \leqslant 3(k-1)-4$, it follows that $|E(M)| \leqslant 3k-4$. $\square$

The family of matroids described next shows that, indeed, the largest members of $\mathcal{N}_k$ have size exactly equal to $3k-4$. For a fixed value of $k \geqslant 4$, a *codex with $k-2$ pages* is any matroid isomorphic to a 3-connected rank-k matroid on the $(3k-4)$-element ground set $\{s_1, s_2\} \cup \{a_i, b_i, c_i\}_{i\in[k-2]}$ where, for each $i \in [k-2]$, the set $\{a_i, b_i, c_i\}$ is a triad, and the sets $\{s_1, a_i, b_i\}$ and $\{s_2, b_i, c_i\}$ are triangles.

One might visualize a codex as $k-2$ *page* planes, each spanned by a triad $\{a_i, b_i, c_i\}$ and joined together at a common *binding* containing s_1 and s_2, subject to the dependence conditions specified (see Fig. 5).

Note that the codices with two pages are the rank-4 wheel and whirl. In these rank-4 cases, there is a particular ambiguity regarding the labeling of elements. The permutation of labels given by $(s_1\ b_1)(s_2\ b_2)(a_2\ c_1)$ yields another labeling satisfying the conditions in the definition. In a codex with more than two pages, there is no ambiguity concerning which pair of elements is in the binding since s_1 and s_2 are the only elements that are in no triads.

Suppose M is a codex with $k-2$ pages. Take $X = \{s_1, s_2\} \cup \{b_i\}_{i\in[k-2]}$. It is evident that no 3-connected proper minor of M contains X in its ground set. Therefore, M and its dual are isomorphic to largest members of $\mathcal{N}_k$. Moreover, by the next lemma, the set X is the only k-element subset of $E(M)$ not contained in a 3-connected proper minor of M unless M is a smallest codex.

5.3. Lemma. *Suppose M is a codex with $k-2$ pages for some $k \geqslant 4$, and $E(M) = \{s_1, s_2\} \cup \{a_i, b_i, c_i\}_{i\in[k-2]}$. Then a k-element subset X of $E(M)$ is not contained in the ground set of a 3-connected proper minor of M if and only if one of the following holds.*

(1) *M is a rank-4 wheel or whirl, and X is the set of spokes or the rim*; *or*
(2) $X = \{s_1, s_2\} \cup \{b_i\}_{i \in [k-2]}$.

Proof. From the remarks above, it suffices to show that if M has a k-element subset X that is not contained in the ground set of a 3-connected proper minor of M, then (1) or (2) holds. First let $k = 4$. Then M is a wheel or a whirl. Write the ground set of M in a fan ordering. Consider the ordering cyclically so that the first and last elements are taken to be consecutive. If two consecutive elements in the ordering are not in X, then some combination of the deletion and contraction of those two elements yields a rank-3 wheel or whirl containing X in its ground set; a contradiction. The fan ordering therefore alternates between X and $E(M) - X$, so X is either the rim of M or the set of spokes of M.

Now suppose that $k > 4$. Note that $\mathrm{si}(M/s_1)$ is not 3-connected as it contains a 2-cocircuit. It follows, by Bixby's Lemma, that $\mathrm{co}(M \backslash s_1)$ is 3-connected. Since, by orthogonality, s_1 is in no triad, this matroid is just $M \backslash s_1$. Hence s_1 is in X. By symmetry, s_2 is in X. If X misses a triad $\{a_i, b_i, c_i\}$ altogether for some $i \in [k-2]$, then the deletion of this triad preserves X and is 3-connected. Therefore, X meets every such triad. Suppose X misses b_i for some $i \in [k-2]$. Then X contains either a_i or c_i but not both. Without loss of generality, suppose X contains a_i. Certainly $\mathrm{co}(M \backslash c_i)$ is not 3-connected since it contains a 2-circuit. Therefore, $\mathrm{si}(M/c_i) \cong M/c_i \backslash b_i$ is 3-connected. This is a contradiction, so $X = \{s_1, s_2\} \cup \{b_i\}_{i \in [k-2]}$. $\square$

Note that if X is the rim of the rank-4 wheel or whirl, then $X = \{a_1, a_2, c_1, c_2\}$. If X is the set of spokes, then $X = \{s_1, s_2, b_1, b_2\}$, the rank-4 case of (2) in the statement of Lemma 5.3. Furthermore, the limitation on X given by the previous lemma is crucial to the proof of the next result, which shows inductively that the codices and their duals are the only largest members of $\mathcal{N}_k$.

5.4. Theorem. *Fix $k \geqslant 4$. Suppose M is a largest member of $\mathcal{N}_k$. Then one of M or M^* is a codex with $k-2$ pages.*

Proof. Theorem 1.3 proves the result when $k = 4$. Suppose $k > 4$ and that the statement holds for $\mathcal{N}_{k-1}$. Now $E(M)$ has a k-subset X that is contained in no 3-connected proper minor of M, and $|E(M)| = 3k - 4$.

Choose e in X. Then M has a minor-minimal 3-connected minor N that uses $X - e$. Thus $N \in \mathcal{N}_{k-1}$, so

$$\left|E(N)\right| \leqslant 3(k-1) - 4 = \left|E(M)\right| - 3 \tag{5.4.1}$$

Now M is a minor-minimal 3-connected matroid that uses e and has N as a minor. Thus, by Theorem 2.3, $|E(M)| \leqslant |E(N)| + 4$. But, when $|E(M)| = |E(N)| + 4$, which arises in (2.3) of that theorem, $M \backslash h/g$ is 3-connected and uses X; a contradiction. Thus

$$\left|E(M)\right| \leqslant \left|E(N)\right| + 3 \tag{5.4.2}$$

Then combining (5.4.1) and (5.4.2) shows that equality holds throughout each, so N is a largest member of $\mathcal{N}_{k-1}$. By the induction assumption, N or N^* is a codex with $k-3$ pages. Moreover, since $|E(M) - E(N)| = 3$, either (3) or (4) of Theorem 2.3 must hold. Assume $M_1 = M$ in that theorem by duality.

Suppose first that (3) holds. If $x \notin X$, then $M/f \backslash x, g$ is 3-connected and uses X; a contradiction. Thus $x \in X$. As $\mathrm{co}(M \backslash g)$ is not 3-connected, $\mathrm{si}(M/g)$ is. If there is no element y of $E(M)$ such that $\{g, x, y\}$ is a triangle, then M/g or $M/g \backslash f$ is 3-connected; a contradiction. If there is such a y but $y \notin X$ then $M/g \backslash y$ or $M/g \backslash f, y$ is 3-connected; a contradiction. Thus $y \in X$. Then interchanging the labels on e and x gives case (4) of Theorem 2.3. Therefore, it suffices to treat that case. By an argument similar to the above, both x and y are in X. Moreover, M has a 5-element fan (x, g, e, f, y)

where $\{g, e, f\}$ is a triad whose deletion from M gives N. The case analysis that follows is structured around the possible identities of x and y in N.

First, consider the case that N is a codex with $k-3$ pages, taking $E(N) = \{s_1, s_2\} \cup \{a_i, b_i, c_i\}_{i\in[k-3]}$. Suppose $k = 5$. Then N is a rank-4 wheel or whirl and, by the previous lemma, $X - e$ is the rim or the set of spokes of N. Suppose first that $X - e$ is the rim $\{a_1, a_2, c_1, c_2\}$ of N. Assume then that $x = a_1$ and $y \in \{a_2, c_2\}$. The unique triad of N containing s_1 is $\{a_1, s_1, a_2\}$. Thus if M has a triad containing s_1, it must be $\{a_1, s_1, a_2\}$. But M has a triangle that meets $E(N)$ in $\{a_1\}$. Hence, by orthogonality, M has no triad containing s_1. Since M also has no triangle that contains s_1 and two elements of X, this is a contradiction to Lemma 2.2.

The case when $k = 5$ and $X - e$ is the set of spokes of N is included in the general argument to follow. Suppose $k > 4$. First observe that if both x and y are in the binding of N, then M is a codex with $k - 2$ pages.

Now suppose that at least one of x and y is not in the binding. Assume then that $x = b_1$ and that $y \in \{s_1, b_2\}$. The following argument shows that $M\backslash a_1$ is 3-connected. Assume the contrary, letting (A, B) be a 2-separation of $M\backslash a_1$. First, observe that the only triad of N containing a_1 is $\{a_1, b_1, c_1\}$. This triad is not a triad of M since M has a triangle that meets $E(N)$ in $\{b_1\}$. Hence M has no triad containing a_1, so $M\backslash a_1$ has no minimal 2-separations. Without loss of generality, assume that $|A \cap \{e, f, g\}| \geqslant 2$. Thus $(A \cup \{e, f, g\}, B - \{e, f, g\})$ is a 2-separation of $M\backslash a_1$, so we may assume that $\{e, f, g\} \subseteq A$. Then A spans $\{x, y\}$, so we may assume that A contains $\{e, f, g, x, y\}$. If $y = s_1$, then $(A \cup a_1, B)$ is a 2-separation of M; a contradiction. Thus $y = b_2$. Note that $(A - \{e, f, g\}, B)$ is a 2-separation of $N\backslash a_1$. But $N\backslash a_1$ is the parallel connection of the triangle $\{b_1, c_1, s_2\}$ and a 3-connected matroid. Hence the only 2-separations of $M\backslash a_1$ have one side equal to $\{b_1, c_1\}$ or $\{b_1, c_1, s_2\}$. Since neither $A - \{e, f, g\}$ nor B is equal to $\{b_1, c_1\}$ or $\{b_1, c_1, s_2\}$, this is a contradiction. Therefore $M\backslash a_1$ is 3-connected, which contradicts the minimality of M.

It remains to consider the case that N is the dual of a codex with $k - 3$ pages. Assume that $k > 5$ since codices with two pages are self-dual. Recall that $M\backslash\{e, f, g\} = N$. Without loss of generality, assume that $x \in \{b_1, s_1\}$. Clearly a_1 is in no triangle of N containing two members of X. Moreover, the only triad of N containing a_1 is $\{a_1, b_1, s_1\}$ and, by orthogonality, it is not a triad of M. Therefore M has no triad containing a_1. This contradiction to Lemma 2.2 completes the proof of the theorem. □

6. Conclusion

The characterizations given by Theorems 1.3 and 1.4 are admittedly not very useful in a computational sense for testing matroid 3-connectivity. A weakening of the equivalent condition to 3-connectivity still gives a characterization and improves the computational expense. Recall that $\mathcal{N}_k$ is the unique minimal set of matroids characterizing 3-connectivity with respect to k-subsets.

6.1. Proposition. *Fix $k \geqslant 4$, and let M be a matroid on at least k elements containing a fixed $(k-2)$-subset Y in its ground set. Then M is 3-connected if and only if, for each pair $\{e, f\} \subseteq E(M) - Y$, there is an $\mathcal{N}_k$-minor of M using $Y \cup \{e, f\}$.*

The proof of this proposition is not difficult using the techniques presented in this paper. The proof is omitted.

It is natural to ask whether there are analogs of the main results of this paper for higher connectivity. While, for example, there must be some minimal set of matroids that characterizes 4-connectivity with respect to k-element subsets for $k \geqslant 6$, there are currently no inductive tools in the style of Bixby's Lemma for 4-connectivity or higher. Therefore the methods used in the proofs of these results cannot be extended to find the appropriate lists.

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